Almost Sure Degrees of Truth and Finite Model Theory of Łukasiewicz Fuzzy Logic

Robert Kosik
Institute of Information Business, Vienna University of Economics and Business Administration, Wirtschaftsuniversität Wien, Augasse 2-6, 1090 Wien, Austria, Europe
kosik@ai.wu-wien.ac.at (Corresponding Author)

and

Christian Fermüller
Institute of Computer Languages, Vienna University of Technology
Technische Universität Wien, Favoritenstraße 9, A-1040 Wien, Austria, Europe
chrisf@logic.at

***Abstract*- An important result of finite model theory are the zero-one laws, which establish that many types of logical formulas hold for either almost all or almost no finite structures. We study the generalization of the zero-one law of first order classical logic to many-valued Łukasiewicz logic. Łukasiewicz logic is a basic fuzzy logic, where truth degrees t are real numbers $0 \leq t \leq 1$. For the special case, where the number of truth values is a power of 2, we prove that to each formula ϕ in first order Łukasiewicz logic there is a (unique) truth value *t* such that ϕ has truth degree *t* almost surely in the sense of finite model theory.**

***Keywords*- almost sure, finite model theory, fuzzy logic, Łukasiewicz logic, zeroone law**

I. ŁUKASIEWICZ LOGIC

According to Zadeh fuzzy logic in the 'narrow' sense is symbolic logic with a comparative notion of truth developed fully in the spirit of classical logic (syntax, semantics, deduction, model theory). One example is Łukasiewicz logic, which is well-developed mathematically. We start by recalling the syntax, semantics and algebra of Łukasiewicz logics.

A. Basic Syntax

Propositional Łukasiewicz logic uses the connectives $\rightarrow$, $\wedge$, $\vee$, &, $\oplus$ and $\leftrightarrow$, the unary connective $\neg$, and the truth constant $\bot$. Formulas are built in the usual way from an infinite set of propositional variables.

In first order Łukasiewicz logic, we have in addition to the connectives also quantifiers $\forall$ and $\exists$. We will consider first-order logic with (crisp, i.e., two-valued) equality = and unary, binary, ternary, etc. relation symbols, but without function symbols or individual constants. Syntactical notions are defined as in the classical first order case.

B. MV Algebras

MV-algebras are for Łukasiewicz logic what Boolean algebras are for classical logic [1]. There is a natural MV-algebra on the set T = [0, 1] of truth degrees. We define a constant $\bot$, a unary operation $\neg$ and binary operations $\rightarrow$, $\wedge$, $\vee$, &, $\oplus$, $\leftrightarrow$ on the unit interval [0, 1] as follows:

$$\neg r = 1 - r \qquad r \oplus s = \min(1, r + s)$$
$$\bot = 0 \qquad r \rightarrow s = (\neg r) \oplus s$$
$$r \wedge s = \min(r, s) \; r \leftrightarrow s = r \rightarrow s \wedge s \rightarrow r$$
$$r \vee s = \max(r, s) \; (r \& s) = \max(0, r + s - 1)$$

Note that all operators can be defined using $\rightarrow$ and the truth value $\bot$. For example, $\neg\varphi$ can be defined as $\varphi \rightarrow \bot$.
By changing the set T of truth values to

$$T = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

we get finite MV algebras, which correspond to finite-valued Łukasiewicz logics.

C. Model Theory

Let σ be a (finite) set of relation symbols, with an "arity" attached to each relation symbol. We call σ the signature.

A 'model' *M* with signature σ is given by a nonempty set *M*, called the domain, together with interpretations R^M of all relation symbols. If *R* is a k-ary relation symbol, then R^M is a map from M^k into *T*.

$\|\varphi\|^{\mathcal{M}} \in [0,1]$ [0, 1] is defined by induction for all closed *M*-formulas in the natural way:

$$\begin{aligned} \|\bot\|^{\mathcal{M}} &= 0 \\ \|\neg\varphi\|^{\mathcal{M}} &= \neg \|\varphi\|^{\mathcal{M}} \\ \|\varphi \oplus \psi\|^{\mathcal{M}} &= \min(1, \|\varphi\|^{\mathcal{M}} + \|\psi\|^{\mathcal{M}}) \\ \|\forall x\, \varphi(x)\|^{\mathcal{M}} &= \inf_{a \in M} \|\varphi(a)\|^{\mathcal{M}} \text{ etc.} \end{aligned}$$

We let $\|\varphi\|$ be the infimum over $\|\varphi\|^M$, taken over all fuzzy models *M*. We say that a closed formula φ is valid iff $\|\varphi\| = 1$, i.e., iff $\|\varphi\|^M = 1$ for all models *M*. It is easy to see that formulas valid in infinite Łukasiewicz logic are also valid in all finite logics.

II. ZERO-ONE LAWS

Finite model theory is a subfield of model theory that focuses on properties of logical languages, such as first-order logic, over finite structures, such as finite groups, graphs, databases, and most abstract machines. It focuses in particular on connections between logical languages and computation, and is closely associated with discrete mathematics, complexity theory, and database theory[3].

An important result of finite model theory are the zero-one laws, which establish that many types of logical formulas hold for either almost all or almost no finite structures.

A. Zero-One Law for First Order Classical Logic

A 'model' *M* is called finite, if its domain *M* is a finite set. If $M = \{1, \ldots, m\}$ we call *M* a labelled finite model with *m* elements.

Let φ be a closed formula in first oder classical logic with equality (without constants and without function symbols). Define $\mu_m(\varphi)$ to be the fraction of labeled models with *m* elements that satisfy the formula φ

$$\mu_m(\varphi) = \frac{\#(\{\mathcal{M} \mid \|\varphi\|^{\mathcal{M}} = 1\})}{\#(\{\mathcal{M}\})} \qquad (1)$$

Here the hash operator denotes cardinality. We say the sentence φ is almost surely true if its asymptotic probability is 1, that is, if $\mu_m(\varphi)$ converges to 1 as *m* goes to infinity.

Theorem 1 (Classical Zero-One-Law): Either φ is true almost surely or $\neg\varphi$ is true almost surely.

A proof can be found in [3]. A quick and very readable introduction is given in [2].

B. Generalization to Łukasiewicz Logic

We will give the generalization of the classical zero-one-law to Łukasiewicz logic. This is our **main theorem**.

Theorem 2 (Almost Sure Degree of Truth): Let φ be a closed formula in first order manyvalued Łukasiewicz logic with equality (without constants and without function symbols). Then there is a (unique) truth value *t* such that $\|\varphi\|^M = t$ is almost surely true.

Our proof relies on an encoding of many-valued models into classical models and a compatible translation of many-valued formulas into two-valued formulas. An abstract and much more general form of this encoding is given in [4].

The encoding of models is given in Section IV. The translation is described in Section V.

The classical zero-one-law is applicable to translated formulas. By looking at the translation of several formulas in parallel we derive a proof of the **main theorem** in Section VI.

III. NORMAL FORMS

These are syntactical tools we will use later for the task of formula translation as described in Equation (5).

A. Existence of Prenex Forms

The following equivalences ("quantifier shifts") known from classical logic are also valid in infinite Łukasiewicz logic. Here, by definition, the bound variable x does not occur free in A:

$$A \rightarrow (\forall x)B \quad \leftrightarrow \quad (\forall x)(A \rightarrow B)$$
$$(\forall x)B \rightarrow A \quad \leftrightarrow \quad (\exists x)(B \rightarrow A)$$
$$A \rightarrow (\exists x)B \quad \leftrightarrow \quad (\exists x)(A \rightarrow B)$$
$$(\exists x)B \rightarrow A \quad \leftrightarrow \quad (\forall x)(B \rightarrow A)$$

Using these equivalences we can rewrite any formula as $(Q\bar{x})P(\bar{x})$ where Q is a chain of quantifiers $(Q_i x_i)$ and P is quantifier free.

B. Disjunctive Normal Form

Let φ be a closed formula of first order n-valued Łukasiewicz logic. By the prenex normal form theorem we have

$$\varphi \leftrightarrow (Q\bar{x})P(\bar{x})$$

Here $P(\bar{x})$ is free of quantifiers, i.e., purely propositional. We look at the atomic expressions $\{P^i\}$ occurring in $P(\bar{x})$, they are relational expressions of the form $R(y_1, \dots, y_k)$ for a k-ary relational symbol R and free variables y_i. Let N denote the number of atoms. An assignment A is a map from the atoms $\{P^i\}$ into the set of truth values. Here equality has to be treated as a classical ({0, 1}-valued) relation. By assigning all possible truth values to the atoms and calculating the value of $P(\bar{x})$ we form the truth table for $P(\bar{x})$. From the truth table we can extract all assignments A which give the truth value t. Let L denote the number of these assignments. We get:

$$\|P(\bar{x})\|^{\mathcal{A}} = t \Leftrightarrow \bigvee_{j=1}^{L} \bigwedge_{i=1}^{N} \|P^i\|^{\mathcal{A}} = t_{ij} \qquad (2)$$

Here the variable P^i runs over all atoms. The variables t_{ij} denote truth values and depend on t on the right. In the classical case our construction corresponds to the disjunctive normal form.

IV. CLASSICAL ENCODING OF MANY-VALUED MODELS

In this section we define an encoding (denoted by ‘tilde’ ˜) of many valued models as classical models. The encoding is bijective, if the number of truth values n is a power of 2.

A. Basic case: one element, a single relation

Consider a many-valued model with a single element and signature $\{R_Ł\}$ (a single relation symbol). A model $M_Ł$ in n-valued logic is given by a single assignment (where the dot denotes the element):

$$R_Ł^{\mathcal{M}}(.) := t = \frac{j}{n-1} \qquad (0 \leq j \leq (n-1))$$

The aim is to give a two-valued representation. The simple idea is to encode a many-valued assignment by a set of two-valued assignments: We write j as a binary number

$$j = \sum_{i=0}^{l-1} a_i\, 2^i \quad \text{with } 0 \leq a_i \leq 1 \qquad (3)$$

and define the classical model $\widetilde{M}$

$$\{\tilde{R}_i^{\tilde{M}}(.) := a_i\} \qquad (4)$$

where $a_i \in \{0, 1\}$ are the digits of j in base 2.

The signature $\widetilde{\sigma}$ of the classical model consists of l ‘clone’ relation symbols $\widetilde{R}_i$.

With this signature, there are 2^l different one-element models. On the other hand using l digits all numbers $0 \leq j \leq (2^l - 1)$ can be represented (leading zeros have to be allowed).

Hence if the number of truth values is a power of 2, i.e., $n = 2^l$, we get a bijection between the set $\{M_Ł\}$ of many-valued models and the set $\{\widetilde{M}\}$ of classical models.

B. Many Elements, Many Relations

So far, our encoding has been defined only in the case of a single element and a single relation. However, extension is trivial: for any k-ary relation symbol R and any k-tuple of elements apply exactly the same encoding, see Equation (3) and Equation (4).

One caveat: as defined, our mapping is based on the bijection between numbers j,

$$\{j | 0 \leq j \leq (2^l - 1)\}$$

and their binary representation as l-tuples

$$\{(a_0, \dots, a_{l-1}) | \bigwedge_{0 \leq i < l} a_i \in \{0, 1\}\}$$

Of course there are other bijections between those two sets. It is possible to apply them to different atoms and combine them.

However, atoms with free variables can only be compatibly translated (see ahead for Equation (8)), if always the same bijection is used. I.e., the encoding of $R(.) = t$ may only depend on t, but not on the argument (.) of R.

V. COMPATIBLE TRANSLATION OF MANY-VALUED FORMULAS

In Section IV-A we have given a bijective encoding of many-valued models $M_Ł$ as classical models $\widetilde{M}$.

The aim of this section is to give a translation of a many-valued formula $\varphi_£$ to a formula $\widetilde{\varphi}^t$ of first order classical logic, which is compatible with the encoding in the following sense:

For all $M_Ł$ and for all truth values t

$$\|\varphi_Ł\|^{\mathcal{M}_Ł} = t \Leftrightarrow \left\|\tilde{\varphi}^t\right\|^{\tilde{\mathcal{M}}}. \quad (5)$$

We write $\widetilde{\varphi}^t$ to indicate that the translation of $\varphi_£$ depends on the truth degree t and on the chosen bijection ˜ between models (but not on a single model $M_Ł$!). In words: If $\varphi_£$ has degree t in $M_Ł$, then its translation $\widetilde{\varphi}^t$ is true in $\widetilde{M}$. Else, if $\varphi_£$ does not have degree t in $M_Ł$, then its translation $\widetilde{\varphi}^t$ is false in $\widetilde{M}$.

A. Translation of Atoms

First, note that equality is a crisp relation, it is not many-valued and the translation is straightforward, as both sets of models share the same domain (in case $t = 0$ negation has to be used on the right).

We now want to map many-valued atomic formulas to classical formulas in a compatible way, i.e.:

$$\|R_Ł(.)\|^{\mathcal{M}_Ł} = t \Leftrightarrow \left\|\tilde{R}^t(.)\right\|^{\tilde{\mathcal{M}}}$$

We have: for each truth value $t = \frac{j}{n-1}$, there is (up to isomorphism) exactly one single-element model $M_Ł$ which makes the left hand side true. By construction the corresponding classical model $\widetilde{M}$ has assignments

$$\{\tilde{R}_i(.) := a_i\}$$

where the a_i are the binary digits of j (possibly including leading zeros).

We want to find a formula $\widetilde{R}^t(.)$ which is only true for these assignments. It is easy to see that

$$\bigwedge_{0\leq i\leq l-1} (\tilde{R}_i(.) \leftrightarrow a_i) \quad (6)$$

fullfills these requirements.

In summary we have ("basic translation"):

$$\|R_Ł(.)\|^{\mathcal{M}_Ł} = \frac{j}{n-1} \Leftrightarrow \bigwedge_{0\leq i\leq l-1} \left\|(\tilde{R}_i(.) \leftrightarrow a_i)\right\|^{\tilde{\mathcal{M}}} \quad (7)$$

with $a_i \in \{0, 1\}$ the first l digits of j in base 2, see Equation (3). (Remark: we could find a compatible translation of atoms for any bijection between truth values and tuples.)

By the construction in Subsection IV-B, the encoding of atoms only depends on the truth value t. Then we have Equation (7) universally:

$$\forall(\bar{x})\|R_Ł(\bar{x})\|^{\mathcal{M}_Ł} = \frac{j}{n-1} \Leftrightarrow \bigwedge_{0\leq i\leq l-1} \left\|(\tilde{R}_i(\bar{x}) \leftrightarrow a_i)\right\|^{\tilde{\mathcal{M}}} \quad (8)$$

Here all relations are k-ary and $\bar{x}$ is shorthand for a list of k free and distinct variables.

B. Compatible Translation of Formulas

It is possible to reduce the translation of "full" formulas to the atomic case by the use of equivalence transformations in Łukasiewicz logic.

Transforming φ to prenex form we get

$$\|(Q\bar{x})P(\bar{x})\|^{\mathcal{M}} = t$$

where Q is a ($\forall$, $\exists$)-chain. The transformation now goes by distinction of cases (formula starts with $\forall$ or $\exists$).

First case: $\|(\forall\chi)\varphi(\chi)\|^M = t$. This is equivalent to:

$$(\exists x)(\|\varphi(x)\|^{\mathcal{M}} = t) \wedge (\forall x)(\|\varphi(x)\|^{\mathcal{M}} \geq t)$$

With only finitely many truth values $t_i \geq t$:

$$\|\varphi(x)\|^{\mathcal{M}} \geq t \Leftrightarrow \bigvee_{t_i \geq t} \|\varphi(x)\|^{\mathcal{M}} = t_i$$

Second case: $\|(\exists\chi)\varphi(\chi)\|^{M} = t$. We get in analogy:

$$(\exists x)(\|\varphi(x)\|^{\mathcal{M}} = t) \wedge (\forall x)(\|\varphi(x)\|^{\mathcal{M}} \leq t)$$

$$\|\varphi(x)\|^{\mathcal{M}} \leq t \Leftrightarrow \bigvee_{t_i \leq t} \|\varphi(x)\|^{\mathcal{M}} = t_i$$

We can move quantifiers outside by repeating these two steps and finally end up with a formula which has expressions $\|P(\overline{\chi})\|^{M} = t$ as its building blocks. These expressions can be syntactically translated by first bringing them to generalized disjunctive form according to Equation (2) and then by compatible translation of the atoms as defined in Equation (8).

VI. THE 'ALMOST SURE DEGREE OF TRUTH'-THEOREM

So far we have constructed an encoding of models and a translation with the following properties: Assume that the number of truth values n is a power of 2. Then for each truth value t and each formula $\varphi_{£}$ of first order n-valued Łukasiewicz logic we can find a formula $\widetilde{\varphi}^{t}$ of classical first order logic, such that for all labelled models $M_Ł$:

$$\|\varphi\|^{\mathcal{M}_Ł} = t \Leftrightarrow \left\|\tilde{\varphi}^{t}\right\|^{\tilde{\mathcal{M}}}.$$

Here $M_Ł$ is a many-valued model, while $\widetilde{M}$ is a classical model. They are linked by a binary encoding of truth values on an atomic level. This encoding is invertible. They have the same domain but different signatures (and different logic). For each cardinality of labels m, the number of models $M_Ł$ is equal to the number of models $\widetilde{M}$.

It is easy to see that $\mu_m(\widetilde{\varphi}^{t})$ (see Equation (1)) is the proportion of labelled many-valued models with truth value t, where m is the number of labels. We have

$$\sum_{i=0}^{n-1} \mu_m(\tilde{\varphi}^{t_i}) = 1 \qquad (9)$$

Applying the classical zero-one theorem to the formula $\widetilde{\varphi}^{t}$, we have for each $t :=$ t_i

$$\lim_{m\to\infty} \mu_m(\tilde{\varphi}^{t_i}) = 0 \vee \lim_{m\to\infty} \mu_m(\tilde{\varphi}^{t_i}) = 1 \quad (10)$$

Exactly one element out of the set of equations

$$\lim_{m\to\infty} \mu_m(\tilde{\varphi}^{t_i}) = 1$$

has to be true, otherwise there is a contradiction with the sum in Equation (9). This proves our main theorem, Theorem 2. Hence every formula of Łukasiewicz logic has an almost sure degree of truth. For $n = 2$ this is the classical zero-one-law.

ACKNOWLEDGEMENT

Robert Kosik was partially supported by the FWF (Austrian Science Foundation) Project no. P16563-N14 and Project no. I143-G15. Christian Fermüller was partially supported by the FWF (Austrian Science Foundation) Project no. I143-G15.

LITERATUR

[1] Petr Hajek. Metamathematics of Fuzzy Logic. Springer Verlag, 1998.

[2] Yuri Gurevich. Zero-One Laws. Bulletin of the European Association for Theoretical Computer Science, No. 51, Feb. 1991, 90–106.

[3] Heinz-Dieter Ebbinghaus und Jörg Flum. Finite Model Theory. Springer Verlag, 1995.

[4] Matthias Baaz and Christian Fermüller. Automated Deduction for Many-Valued Logics. Handbook of Automated Reasoning, Elsevier Science Publishers, 2001, 1357–1402.